

# A GEOMETRIC MEASURE-TYPE REGULARITY CRITERION FOR SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS

Z. GRUJIĆ

ABSTRACT. A local anisotropic geometric measure-type condition on the super-level sets of solutions to the 3D NSE preventing the formation of a finite-time singularity is presented; essentially, *local one-dimensional sparseness* of the regions of intense fluid activity in a very weak sense.

## 1. PROLOGUE

Rigorous study of geometric depletion of the nonlinearity in the 3D Navier-Stokes equations (3D NSE) was initiated by Constantin in [Co94]; the approach was based on the singular integral representation formula for the stretching factor in the evolution of the vorticity magnitude featuring a geometric kernel depleted by coherence of the vorticity direction. This representation was subsequently utilized by Constantin and Fefferman in [CoFe93] to show that as long as the vorticity direction is Lipschitz-coherent, no finite-time blow up can occur and later by Beirao da Veiga and Berselli in [daVeigaBe02], where the Lipschitz-coherence regularity condition was replaced with  $\frac{1}{2}$ -Hölder.

Spatiotemporal localization of the  $\frac{1}{2}$ -Hölder-coherence regularity criterion was performed in [GrZh06, Gr09], and also – utilizing a different localization procedure – by Chae, Kang and Lee in [ChKaLe07].

The  $\frac{1}{2}$ -Hölder-coherence condition is super-critical with respect to the natural scaling of the 3D NSE; a family of scaling-invariant, critical, hybrid geometric-analytic local regularity criteria – including a scaling-invariant improvement of the  $\frac{1}{2}$ -Hölder-coherence condition – was presented in [GrGu10-1].

In the realm of the mathematical theory of turbulence, the  $\frac{1}{2}$ -Hölder-coherence condition was recently ([DaGr11-1]) paired with the condition on a modified Kraichnan scale to obtain a first rigorous evidence of existence of (anisotropic) enstrophy cascade in 3D viscous incompressible flows.

A different approach to discovering geometric scenarios ruling out formation of singularities in the 3D NSE was introduced in [Gr01]. The main idea was to utilize the local-in-time spatial analyticity properties of solutions in  $L^p$  ([GrKu98]) via the plurisubharmonic measure maximum principle – a generalization to  $\mathbb{C}^n$  (cf. [Sad81]) of the classical harmonic measure majorization principle in the complex plane (the log-convexity of the modulus of an analytic function; see, e.g., [Nev70]).

The regularity criterion derived in [Gr01] is a condition on the regions of intense fluid activity near a possible blow up time requiring local existence of a sparse coordinate *projection* on the scale comparable to the uniform radius of spatial analyticity. The estimate on the plurisubharmonic measure was performed within the framework of product-type domains – hence the requirement on a coordinate projection. This could be somewhat relaxed, but not substantially due to the rigidity of the  $\mathbb{C}^n$  structure. Also, once the computation of the plurisubharmonic measure was reduced to the computation of the individual (coordinate) harmonic measures, the estimate on the harmonic measure was carried out with respect to an infinite strip, giving the argument a nonlocal character.

In the present work, we completely bypass the rigidity of the  $\mathbb{C}^n$  structure, resulting in a much weaker *local geometric measure-type condition*. Utilizing translational and rotational invariance of the 3D NSE, as well as some basic geometric properties of the harmonic measure, the argument is ultimately reduced to the problem of estimating the harmonic measure of an arbitrary closed subset of  $[-1, 1]$  computed at 0 with respect to the unit disk. This is a generalization of the classical Beurling's problem [Beu33, Nev70] proposed by Segawa in [Seg88]; a symmetric version was solved by Essen and Haliste in [EssHa89], and the general case relatively recently by Solynin in [Sol99] via a general symmetrization

argument. The utility of Solynin's result here is that it allows us to formulate the condition in view in terms of the ratio of one-dimensional Lebesgue measures, rather than in terms of a specific 'rearrangement' of the one-dimensional trace of the region of intense fluid activity. The engine behind the proof is the interplay between the diffusion in the model – quantified by the local-in-time (sharp) analytic smoothing in  $L^\infty$  – and the geometric properties of the harmonic measure.

A precise statement of our regularity criterion will be given in the main text; at this point, we convey the essence of the result. Denoting the region of intense fluid activity at time  $s$  by  $\Omega_s(M)$  and the lower bound on the uniform radius of spatial analyticity by  $\rho(s)$  (this takes place near a potential singular time), the condition in view is simply a stipulation that for a given point  $x_0$ , there exists a radius  $r = r(x_0)$ ,  $0 < r \leq \rho(s)$  and a unit vector  $d = d(x_0)$ , such that

$$(1.1) \quad \frac{|\Omega_s(M) \cap (x_0 - rd, x_0 + rd)|}{2r} \leq \delta$$

for some  $\delta$  in  $(0, 1)$ . (It will transpire that it is enough to require (1.1) on a suitably chosen finite sequence of times.) There are two versions of the result, one for the velocity and one for the vorticity, based on the spatial analyticity estimates on solutions to the velocity and vorticity formulations of the 3D NSE, respectively.

It is plain that (1.1) is a much weaker condition than the one in [Gr01]; all that is needed here is local sparseness of a one-dimensional *trace* of the region of intense fluid activity in a very weak sense. On the other hand, it is of a different nature and hence not directly comparable to the coherence of the vorticity direction-type regularity criteria.

The main efficacy of the aforementioned regularity criterion is in ruling out various sparse geometric scenarios for a finite-time blow up, both in the velocity and the vorticity formulations. As succinctly put by P. Constantin, "intermittency implies regularity" [PC11]. An example of interest that can be ruled out is a blow up scenario in which the region of intense vorticity (defined as a region in which the vorticity magnitude exceeds a suitable fraction of the  $L^\infty$ -norm) is – at suitable near-blow up times – comprised of vortex filaments with diameters of the cross-sections bounded above by the uniform radius of spatial analyticity.

The paper is organized as follows. In Section 2, we collect relevant properties of the harmonic measure in the plane, and in Section 3 we recall the local-in-time spatial analyticity of solutions in  $L^\infty$ . Section 4 contains the main result. The last section indicates a scenario which – in a statistically significant sense – leads to closing the scaling gap in the regularity problem.

## 2. HARMONIC MEASURE

Basic properties of the harmonic measure in the complex plane can be found, e.g., in [Nev70, Ahl10]. First, we briefly recall a few relevant facts following [Ahl10].

Let  $\Omega$  and  $K$  be an open and a closed set in the complex plane, respectively. When the geometry of  $\Omega \setminus K$  is not too convoluted, there exists a unique bounded harmonic function on  $\Omega \setminus K$ , denoted by  $\omega = \omega(\cdot, \Omega, K)$ , such that – in the sense of a well-defined limit as a point approaches the boundary –  $\omega$  is equal to 1 on  $K$

and 0 on the rest of the boundary;  $\omega(z, \Omega, K)$  is the harmonic measure of  $K$  with respect to  $\Omega$  computed at  $z$ .

Two straightforward consequences of the general harmonic measure majorization principle, c.f. Theorem 3.1 [Ahl10], are the following ([Ahl10], p. 39).

**Proposition 2.1.** *The harmonic measure is increasing (as a measure) with respect to both  $K$  and  $\Omega$ .*

**Proposition 2.2.** *Let  $f$  be analytic in  $\Omega \setminus K$ ,  $|f| \leq M$ , and  $|f| \leq m$  on  $K$  (in the sense of  $\limsup$  as a point approaches the boundary). Then*

$$|f(z)| \leq m^\theta M^{1-\theta}$$

for any  $z$  in  $\Omega \setminus K$ , where  $\theta = \omega(z, \Omega, K)$ .

This is a refined form of the maximum modulus principle for analytic functions in  $\Omega \setminus K$  (the log-convexity of the modulus of  $f$  – sometimes referred to as “two-constants theorem”).

Another useful property of the harmonic measure is the following (see., eg., [Nev70]).

**Proposition 2.3.** *The harmonic measure is invariant with respect to conformal mappings.*

Finally, we recall a result on extremal properties of the harmonic measure in the unit disk  $\mathbb{D}$  obtained by Solynin in [Sol99].

**Theorem 2.1.** *Let  $K$  be a closed subset of  $[-1, 1]$  such that  $|K| = 2\lambda$  for some  $\lambda$ ,  $0 < \lambda < 1$ , and suppose that  $0 \in \mathbb{D} \setminus K$ . Then*

$$\omega(0, \mathbb{D}, K) \geq \omega(0, \mathbb{D}, K_\lambda) = \frac{2}{\pi} \arcsin \frac{1 - (1 - \lambda)^2}{1 + (1 - \lambda)^2}$$

where  $K_\lambda = [-1, -1 + \lambda] \cup [1 - \lambda, 1]$ .

The above theorem provides a generalization of the classical Beurling’s result [Beu33] in which  $K$  is a finite union of intervals lying on one side of the origin. This was conjectured by Segawa in [Seg88], and the symmetric version was previously resolved in [EssHa89].

### 3. SPATIAL ANALYTICITY IN $L^\infty$

The 3D NSE equations read

$$(3.1) \quad u_t + (u \cdot \nabla)u = -\nabla p + \Delta u$$

supplemented with the incompressibility condition  $\nabla u = 0$ , where  $u$  is the velocity of the fluid and  $p$  the pressure (the viscosity is set to 1).

A method for deriving explicit local-in-time lower bounds on the uniform radius of spatial analyticity of solutions to the NSE in  $L^p$  was introduced in [GrKu98]; see also [Ku99] for analogous results in the vorticity formulation. We will make use of the following sharp analyticity estimate in  $L^\infty$  (cf. [Gu10]; [Ku03] for the corresponding real result).

**Theorem 3.1.** *Let  $u_0$  be in  $L^\infty(\mathbb{R}^3)$ . Then, there exists an absolute constant  $c_0 > 1$  such that setting  $T = \frac{1}{c_0^2 \|u_0\|_\infty^2}$ , a unique mild solution  $u = u(t)$  on  $[0, T]$  has the analytic extension  $U = U(t)$  to the region*

$$\mathcal{R}_t = \{x + iy \in \mathbb{C}^3 : |y| \leq \frac{1}{c_0} \sqrt{t}\}$$

for any  $t$  in  $(0, T]$ . In addition,

$$\|U(t)\|_{L^\infty(\mathcal{R}_t)} \leq c_0 \|u_0\|_\infty$$

for all  $t$  in  $[0, T]$ .

Recall that the vorticity formulation of the 3D NSE reads

$$(3.2) \quad \omega_t + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u + \triangle \omega$$

where  $\omega = \text{curl } u$  is the vorticity.

The vorticity version of the above theorem is as follows (the proof is analogous; utilizing the Biot-Savart law to close each iteration).

**Theorem 3.2.** *Let  $\omega_0$  be in  $L^\infty(\mathbb{R}^3)$ . Then, there exists an absolute constant  $d_0 > 1$  such that setting  $T = \frac{1}{d_0^2 \|\omega_0\|_\infty^2}$ , a unique mild solution  $\omega = \omega(t)$  on  $[0, T]$  has the analytic extension  $\Omega = \Omega(t)$  to the region*

$$\mathcal{R}_t = \{x + iy \in \mathbb{C}^3 : |y| \leq \frac{1}{d_0} \sqrt{t}\}$$

for any  $t$  in  $(0, T]$ . In addition,

$$\|\Omega(t)\|_{L^\infty(\mathcal{R}_t)} \leq d_0 \|\omega_0\|_\infty$$

for all  $t$  in  $[0, T]$ .

#### 4. THE MAIN RESULT

We start with introducing a geometric measure-theoretic concept of *weak local linear sparseness of a set around a point, at a given scale*, suitable for our purposes.

**Definition 4.1.** *Let  $x_0$  be a point in  $\mathbb{R}^3$ ,  $r > 0$ ,  $S$  an open subset of  $\mathbb{R}^3$  and  $\delta$  in  $(0, 1)$ .*

*The set  $S$  is linearly  $\delta$ -sparse around  $x_0$  at scale  $r$  in weak sense if there exists a unit vector  $d$  in  $S^2$  such that*

$$\frac{|S \cap (x_0 - rd, x_0 + rd)|}{2r} \leq \delta.$$

In what follows, we derive the main result for the velocity formulation and simply state the analogous result for the vorticity formulation; modifying the proof in the second case is essentially relabeling.

For  $M > 0$ , denote by  $\Omega_t(M)$  the super-level set at time  $t$ ; more precisely,

$$\Omega_t(M) = \{x \in \mathbb{R}^3 : |u(x, t)| > M\}.$$

Then, our main result reads as follows.

**Theorem 4.1.** *Suppose that a solution  $u$  is regular on an interval  $(0, T^*)$ . (Recall that  $u$  is then necessarily in  $C((0, T^*); L^\infty)$ .)*

*Fix  $\delta$  in  $(0, 1)$ , and let  $h = h(\delta) = \frac{2}{\pi} \arcsin \frac{1-\delta^2}{1+\delta^2}$  and  $\alpha = \alpha(\delta) \geq \frac{1-h}{h}$ . Assume that there exists  $\epsilon > 0$  such that for any  $t$  in  $(T^* - \epsilon, T)$ , either*

- (i)  $t + \frac{1}{c_0^2 \|u(t)\|_\infty^2} \geq T^*$  ( $c_0$  is the constant featured in Theorem 3.1), or
- (ii) *there exists  $s = s(t)$  in  $\left[t + \frac{1}{4c_0^2 \|u(t)\|_\infty^2}, t + \frac{1}{c_0^2 \|u(t)\|_\infty^2}\right]$  such that for any spatial point  $x_0$ , there exists a scale  $r$ ,  $0 < r \leq \frac{1}{2c_0^2 \|u(t)\|_\infty}$ , with the property that the super-level set  $\Omega_s(M)$  is linearly  $\delta$ -sparse around  $x_0$  at scale  $r$  in weak sense; here,  $M = M(\delta) = \frac{1}{c_0^2} \|u(t)\|_\infty$ .*

*Then, there exists  $\gamma > 0$  such that  $u$  is in  $L^\infty((T^* - \epsilon, T^* + \gamma); L^\infty)$ , i.e.,  $T^*$  is not a singular time.*

*Proof.* There are two cases to consider.

Case I (i) holds for some  $t$  in  $(T^* - \epsilon, T^*)$ .

In this case, the statement of the theorem follows from Theorem 3.1, setting the initial time to  $t$ .

Case II (ii) holds for all  $t$  in  $(T^* - \epsilon, T^*)$ .

Pick a time  $t_0$  in  $(T^* - \epsilon, T^*)$ , and let  $\delta, h, \alpha, M$  and  $s_0 = s(t_0)$  be as in the statement of the theorem. Then, for any  $x_0$  in  $\mathbb{R}^3$ , there exists an  $r = r(x_0)$ ,  $0 < r \leq \frac{1}{2c_0^2 \|u(t_0)\|_\infty}$ , and a direction vector  $d = d(x_0)$ , such that

$$\frac{|\Omega_{s_0}(M) \cap (x_0 - rd, x_0 + rd)|}{2r} \leq \delta.$$

The intent is to show

$$(4.1) \quad \|u(s_0)\|_\infty \leq \|u(t_0)\|_\infty.$$

Fix  $x_0$ . Recall that the NSE exhibit translational and rotational invariance in the spatial variable. Translate for  $-x_0$ , rotate by the matrix  $Q$  transforming the unit direction  $d$  to the coordinate vector  $e_1$  and denote the transformed solution by  $u_{x_0, Q}$ . Then,  $u_{x_0, Q}(x, t) = Qu(Q^{-1}(x + x_0), t)$ .

Solve the NSE locally-in-time starting at  $t_0$ ; the spatial analyticity properties of the solution at time  $s_0$  are given by simply translating in time the statement of Theorem 3.1.

Moreover, since the rotation  $Q$  has no effect on computing the norms, the transformed solution  $u_{x_0, Q}$  at time  $s_0$  enjoys exactly the same analyticity features.

In particular – focusing on the first coordinate –  $u_{x_0, Q}(s_0)$  is spatially analytic on a strip, symmetric around the real axis, with the width equal to (at least)

$$\rho(s_0) = \frac{1}{c_0^2 \|u(t_0)\|_\infty}.$$

The region of interest is the disk around the origin with the radius  $r$ ,  $D_r$ . Note that  $D_r$  is contained in the domain of analyticity of  $u_{x_0, Q}(s_0)$ .

Our goal is to obtain an improved estimate on  $u_{x_0,Q}(0, s_0)$ . Denote by  $K$  the complement of the image of the set  $\Omega_{s_0}(M) \cap (x_0 - rd, x_0 + rd)$ , under the change of coordinates, in  $[-r, r]$ . Then,  $K$  is closed, and the sparseness assumption implies  $|K| \geq 2r(1 - \delta)$ . If  $0 \in K$ ,  $|u_{x_0,Q}(0, s_0)| < \|u(t_0)\|_\infty$ , and we are done (with this  $x_0$ ). If not, the harmonic measure maximum principle – Proposition 2.2 – together with the  $L^\infty$ -bound on the complexified solution stated in Theorem 3.1, implies

$$(4.2) \quad |u_{x_0,Q}(0, s_0)| \leq \left( \frac{1}{c_0^\alpha} \|u(t_0)\|_\infty \right)^{\omega(0, D_r, K)} \left( c_0 \|u(t_0)\|_\infty \right)^{1-\omega(0, D_r, K)}.$$

Recall that the harmonic measure is invariant under conformal mappings (Proposition 2.3). In particular, it is invariant under the scaling map  $z \mapsto \frac{1}{r}z$ . This paired with the monotonicity of the harmonic measure with respect to  $K$  (Proposition 2.1) and Theorem 2.1 yields

$$(4.3) \quad \omega(0, D_r, K) \geq \frac{2}{\pi} \arcsin \frac{1 - \delta^2}{1 + \delta^2} = h.$$

Combining the estimates (4.2) and (4.3) leads to

$$(4.4) \quad |u_{x_0,Q}(0, s_0)| \leq \left( \frac{1}{c_0^\alpha} \|u(t_0)\|_\infty \right)^h \left( c_0 \|u(t_0)\|_\infty \right)^{1-h} \leq \|u(t_0)\|_\infty.$$

This, in turn, implies  $|u(x_0, s_0)| \leq \|u(t_0)\|_\infty$ , and since  $x_0$  was an arbitrary spatial point in  $\mathbb{R}^3$ ,  $\|u(s_0)\|_\infty \leq \|u(t_0)\|_\infty$ .

Let

$$M_0 = \|u(t_0)\|_\infty.$$

Setting  $t_1 = s_0$  and repeating the argument yields  $\|u(s_1)\|_\infty \leq \|u(t_1)\|_\infty \leq M_0$ , where  $s_1 = s(t_1)$ . After finitely many steps, we reach the time  $s_n, s_n < T^*$  such that  $\|u(s_n)\|_\infty \leq M_0$  and  $s_n + \frac{1}{c_0^2 M_0} > T^*$ . The statement of the theorem now follows from Theorem 3.1, setting the initial time to  $s_n$ .  $\square$

**Remark 4.1.** It is plain from the proof that it is enough to assume the condition on a *finitely many* suitably chosen times.

The vorticity version and the proof are completely analogous – utilizing Theorem 3.2 in place of Theorem 3.1.

For  $M > 0$ , denote by  $\Omega_t^\omega(M)$  the vorticity super-level set at time  $t$ ; more precisely,

$$\Omega_t^\omega(M) = \{x \in \mathbb{R}^3 : |\omega(x, t)| > M\}.$$

**Theorem 4.2.** *Suppose that a solution  $u$  is regular on an interval  $(0, T^*)$ .*

*Fix  $\delta$  in  $(0, 1)$ , and let  $h = h(\delta) = \frac{2}{\pi} \arcsin \frac{1-\delta^2}{1+\delta^2}$  and  $\alpha = \alpha(\delta) \geq \frac{1-h}{h}$ . Assume that there exists  $\epsilon > 0$  such that for any  $t$  in  $(T^* - \epsilon, T)$ , either*

$$(i) \quad t + \frac{1}{d_0^2 \|\omega(t)\|_\infty} \geq T^* \quad (d_0 \text{ is the constant featured in Theorem 3.2}), \text{ or}$$

*(ii) there exists  $s = s(t)$  in  $\left[t + \frac{1}{4d_0^2 \|\omega(t)\|_\infty}, t + \frac{1}{d_0^2 \|\omega(t)\|_\infty}\right]$  such that for any spatial point  $x_0$ , there exists a scale  $r$ ,  $0 < r \leq \frac{1}{2d_0^2 \|\omega(t)\|_\infty^{\frac{1}{2}}}$ , with the property that the super-level set  $\Omega_s^\omega(M)$  is linearly  $\delta$ -sparse around  $x_0$  at scale  $r$  in weak sense; here,  $M = M(\delta) = \frac{1}{d_0^\alpha} \|\omega(t)\|_\infty$ .*

*Then, there exists  $\gamma > 0$  such that  $\omega$  is in  $L^\infty((T^* - \epsilon, T^* + \gamma); L^\infty)$ , i.e.,  $T^*$  is not a singular time.*

## 5. EPILOGUE

Direct numerical simulations of turbulent flows reveal (see, e.g., [SJO91]) that the preferred geometric signature of the regions of intense vorticity is the one of vortex filaments. The general agreement seems to be that the length of a filament is – in a statistically significant sense – comparable with the macro scale, while the scaling of the diameter of the filament’s cross-section seems to be harder to pin down (although, it is mostly found to be comparable to some version of Kolmogorov dissipation scale). For rigorous mathematical results concerning creation and dynamics of vortex tubes in turbulent flows, the reader is referred to [CPS95].

Let us for a moment adopt the aforementioned geometry as a blow up scenario, and define the region of intense vorticity at a near-blow up time  $t$  to be the region in which the vorticity magnitude exceeds a fraction of  $\|\omega(t)\|_\infty$ . Then, Theorem 4.2 implies that as long as the diameters of the filaments’ cross-sections are dominated by  $\frac{1}{C_1} \frac{1}{\|\omega(t)\|_\infty^{\frac{1}{2}}}$ , for a suitable constant  $C_1 > 1$ , no blow up can occur. At this point, recall that starting with the initial vorticity a finite Radon measure, the  $L^1$ -norm of the vorticity is bounded – uniformly in time – over any interval  $(0, T)$  [Co90]. Tchebyshev then implies the decrease of the distribution function of the vorticity of at least  $\frac{1}{\beta}$ ; consequently, the volume of the region of intense vorticity decreases at least as  $C_2 \frac{1}{\|\omega(t)\|_\infty}$ . Assuming that the length of a filament is comparable with the macro scale, this implies the decrease of the diameters of the filaments’ cross-sections of at least  $C_3 \frac{1}{\|\omega(t)\|_\infty^{\frac{1}{2}}}$ , which is exactly the scale needed for the application of Theorem 4.2.

The above ruminations offer a geometric scenario leading to closing the scaling gap in the regularity problem. Assuming that the ‘shape’, i.e., the general geometry is correct, the weakest link is the assumption that the length of a filament be comparable to the macro scale; this was simply borrowed from the picture painted by the numerical simulations. However, in a very recent work [DaGr11-2], the authors utilized the *ensemble averaging process* introduced in their study of turbulent



cascades in physical scales of 3D incompressible flows ([DaGr10]) to arrive at preliminary results indicating that the averaged vortex-stretching term is – near a possible blow up time  $T^*$  – positive across a range of scales extending from a power of a modified Kraichnan scale to the macro scale. This provides a mathematical evidence of persistence of the macro scale-long vortex filaments (in a statistically significant sense), and the pertaining research will be pursued in the future.

**ACKNOWLEDGMENTS** The author thanks Professor Peter Constantin for inspiring discussions and Department of Mathematics at the University of Chicago for hospitality while being a Long Term Visitor in Fall 2011; the support of the *Research Council of Norway* via the grant number 213473 - FRINATEK is gratefully acknowledged.

#### REFERENCES

- [Beu33] A. Beurling, These, Uppsala, 1933.
- [Le34] J. Leray, Acta Math. **63**, 193 (1934).
- [Nev70] R. Nevanlinna, Analytic functions, Springer-Verlag, 1970 (Translated from the Second German Edition).
- [Sad81] A. Sadullaev, Russian Math. Surveys **36**, 61 (1981).
- [Seg88] S. Segawa, Proc. Am. Math. Soc. **103**, 177 (1988).
- [EssHa89] M. Essen and K. Haliste, Complex Variables **12**, 137 (1989).
- [Co90] P. Constantin, Comm. Math. Phys. **129**, 241 (1990).
- [SJO91] Z.-S. She, E. Jackson and S. Orszag, Proc. R. Soc. Lond. A **434**, 101 (1991).
- [Co94] P. Constantin, SIAM Rev. **36**, 73 (1994).
- [CoFe93] P. Constantin and C. Fefferman, Indiana Univ. Math. J. **42**, 775 (1993).
- [CPS95] P. Constantin, I. Procaccia and D. Segel, Phys. Rev E **51**, 3207 (1995).
- [GrKu98] Z. Grujić and I. Kukavica, J. Funct. Anal. **152**, 447 (1998).
- [Ku99] I. Kukavica, Indiana Univ. Math. J. **48**, 1057 (1999).
- [Sol99] A. Yu. Solynin, Journal of Mathematical Sciences **95**, 2256 (1999).
- [Gr01] Z. Grujić, Indiana Univ. Math. J. **50**, 1309 (2001).
- [daVeigaBe02] H. Beirao da Veiga and L.C. Berselli, Diff. Int. Eqs. **15**, 345 (2002).
- [Ku03] I. Kukavica, J. Diff. Equations **194**, 39 (2003).
- [GrZh06] Z. Grujić and Qi Zhang, Comm. Math. Phys. **262**, 555 (2006).
- [ChKaLe07] D. Chae, K. Kang and J. Lee, Comm. PDE **32**, 1189 (2007).
- [Gr09] Z. Grujić, Comm. Math. Phys. **290**, 861 (2009).
- [Ahl10] L.V. Ahlfors, Conformal invariants: topics in geometric function theory, AMS Chelsea Pub., 2010.
- [GrGu10-1] Z. Grujić and R. Guberović, Comm. Math. Phys. **298**, 407 (2010).
- [Gu10] R. Guberović, Discrete Cont. Dynamical Systems **27**, 231 (2010).
- [DaGr10] R. Dascaliuc and Z. Grujić, Comm. Math. Phys. **305**, 199 (2011).
- [DaGr11-1] R. Dascaliuc and Z. Grujić, Comm. Math. Phys. (submitted) (2012).
- [DaGr11-2] R. Dascaliuc and Z. Grujić (in preparation) (2012).
- [PC11] P. Constantin, private communication (2011).